

A NOTE ON CRYSTALLINE LIFTINGS IN THE \mathbb{Q}_p CASE

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ABSTRACT. Let $\rho : G_{\mathbb{Q}_p} \rightarrow \mathrm{GL}_d(\mathcal{O}_{\overline{\mathbb{Q}_p}})$ be a crystalline representation with Hodge-Tate weights in $[0, p]$, such that its reduction $\overline{\rho}$ is upper triangular. Under certain conditions, we prove that $\overline{\rho}$ has an upper triangular crystalline lift ρ' such that $\mathrm{HT}(\rho') = \mathrm{HT}(\rho)$. The method is based on the author's previous work, combined with an inspiration from the work of Breuil-Herzig.

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1. INTRODUCTION

1.1. **Overview.** Let $G_{\mathbb{Q}_p} := \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ be the Galois group of \mathbb{Q}_p with respect to a fixed algebraic closure $\overline{\mathbb{Q}_p}$. Let E/\mathbb{Q}_p be a finite extension, \mathcal{O}_E the ring of integers, ω_E a fixed uniformizer, and $k_E = \mathcal{O}_E/\omega_E\mathcal{O}_E$ the residue field. We will use the following notations often, **(CRYS)**:

- Let $p > 2$ be an odd prime. Suppose V is a crystalline representation of $G_{\mathbb{Q}_p}$ of E -dimension d , such that the Hodge-Tate weights $\mathrm{HT}(V) = \{0 = r_1 < \dots < r_d \leq p\}$.
- Let $\rho = T$ be a $G_{\mathbb{Q}_p}$ -stable \mathcal{O}_E -lattice in V , and $\hat{\mathfrak{M}} \in \mathrm{Mod}_{\mathfrak{S}_{\mathcal{O}_E}}^{\varphi, \hat{G}}$ the (φ, \hat{G}) -module (with \mathcal{O}_E -coefficient) attached to T . Let $\overline{\rho} := T/\omega_E T$ be the reduction. Let $\overline{\hat{\mathfrak{M}}}$ be the reduction of $\hat{\mathfrak{M}}$, and let $\overline{\mathfrak{M}}$ be the reduction of \mathfrak{M} .

1.1.1. **Theorem.** *With notations in (CRYS). Suppose that $\overline{\rho}$ is upper triangular, i.e., there exists an increasing filtration $0 = \mathrm{Fil}^0 \overline{\rho} \subset \mathrm{Fil}^1 \overline{\rho} \subset \dots \subset \mathrm{Fil}^d \overline{\rho} = \overline{\rho}$ such that $\mathrm{Fil}^i \overline{\rho} / \mathrm{Fil}^{i-1} \overline{\rho} = \overline{\chi}_{d-i}$, $\forall 1 \leq i \leq d$, where $\overline{\chi}_i$ are some characters. Suppose $\overline{\chi}_i \overline{\chi}_j^{-1} \neq \overline{\varepsilon}_p$, $\forall i \neq j$ where $\overline{\varepsilon}_p$ is the reduction of the cyclotomic character. Then there exists an upper triangular crystalline representation ρ' such that $\overline{\rho'} \cong \overline{\rho}$, and $\mathrm{HT}(\rho') = \mathrm{HT}(\rho)$ as sets.*

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Theorem 1.1.1 strengthens [Gao15, Cor. 0.2(1)] in the \mathbb{Q}_p -case, and of course have direct application to weight part of Serre's conjectures as in *loc. cit.*. In our Theorem 1.1.1, we do *not* require the Condition (C-1) of [Gao15, §3], and only require a weaker version of Condition (C-2A) of [Gao15, §6] (The Condition (C-2B) of *loc. cit.* in general will not be satisfied in our current paper.)

The proof of our theorem still uses results in [Gao15] to study the possible shape of upper triangular reductions of crystalline representations. The difference in the current paper is a different crystalline lifting technique, which is inspired by some group theory developed in [BH15]. Roughly speaking, we can use the group theory to conjugate our upper triangular $\bar{\rho}$ to another upper triangular form, which can be lifted to an *ordinary* (in particular, upper triangular) crystalline representation via the result of [GG12]. The lifting process via *loc. cit.* is in some sense easier than those used in [Gao15] (which is generalization of methods in [GLS14, GLS15]). However, we can only apply this technique in the \mathbb{Q}_p -case, because it seems that we cannot apply the group theory in [BH15] to deal with general K/\mathbb{Q}_p case for our problem. Let us also remark that our current paper shows a much refined structure for upper triangular reductions of crystalline representations.

The paper is organized as follows. In Section 2, we review the theory of Kisin modules and (φ, \hat{G}) -modules with \mathcal{O}_E -coefficients. In Section 3, we review the group theory in [BH15]. In Section 4, we study the shape of upper triangular torsion (φ, \hat{G}) -modules, using results in [Gao15], as well as techniques inspired by the group theory in Section 3. Finally in Section 5, we prove our crystalline lifting theorem.

1.2. Notations. The notations in the following are taken directly from [Gao15]. In particular, they are valid for any finite extension K/\mathbb{Q}_p (and we use K_0 to denote the maximal unramified sub-extension of K , and k the residue field of K). See *loc. cit.* for any unfamiliar terms and more details.

In this paper, we sometimes use boldface letters (e.g., \mathbf{e}) to mean a sequence of objects (e.g., $\mathbf{e} = (e_1, \dots, e_d)$ a basis of some module). We use $\text{Mat}(?)$ to mean the set of matrices with elements in $?$. We use notations like $[u^{r_1}, \dots, u^{r_d}]$ to mean a diagonal matrix with the diagonal elements in the bracket. We use Id to mean the identity matrix. For a matrix A , we use $\text{diag} A$ to mean the diagonal matrix formed by the diagonal of A .

In this paper, **upper triangular** always means successive extension of rank-1 objects. We use notations like $\mathcal{E}(m_d, \dots, m_1)$ (note the order of objects) to mean the set of all upper triangular extensions of rank-1 objects in certain categories. That is, m is in $\mathcal{E}(m_d, \dots, m_1)$ if there is an increasing filtration $0 = \text{Fil}^0 m \subset \text{Fil}^1 m \subset \dots \subset \text{Fil}^d m = m$ such that $\text{Fil}^i m / \text{Fil}^{i-1} m = m_i, \forall 1 \leq i \leq d$.

We normalize the Hodge-Tate weights so that $\text{HT}_\kappa(\varepsilon_p) = 1$ for any $\kappa : K \rightarrow \overline{\mathbb{Q}_p}$, where ε_p is the p -adic cyclotomic character.

We fix a system of elements $\{\pi_n\}_{n=0}^\infty$ in \overline{K} , where $\pi_0 = \pi$ is a uniformizer of K , and $\pi_{n+1}^p = \pi_n, \forall n$. Let $K_n = K(\pi_n)$, $K_\infty = \cup_{n=0}^\infty K(\pi_n)$, and $G_\infty := \text{Gal}(\overline{K}/K_\infty)$.

We fix a system of elements $\{\mu_{p^n}\}_{n=0}^\infty$ in \overline{K} , where $\mu_1 = 1$, μ_p is a primitive p -th root of unity, and $\mu_{p^{n+1}}^p = \mu_{p^n}, \forall n$. Let

$$K_{p^\infty} = \cup_{n=0}^\infty K(\mu_{p^n}), \quad \hat{K} = K_{\infty, p^\infty} = \cup_{n=0}^\infty K(\pi_n, \mu_{p^n}).$$

Note that \hat{K} is the Galois closure of K_∞ , and let

$$\hat{G} = \text{Gal}(\hat{K}/K), \quad H_K = \text{Gal}(\hat{K}/K_\infty), \quad G_{p^\infty} = \text{Gal}(\hat{K}/K_{p^\infty}).$$

When $p > 2$, then $\hat{G} \simeq G_{p^\infty} \rtimes H_K$ and $G_{p^\infty} \simeq \mathbb{Z}_p(1)$, and so we can (and do) fix a topological generator τ of G_{p^∞} . And we can furthermore assume that $\mu_{p^n} = \frac{\tau(\pi_n)}{\pi_n}$ for all n .

Let $C = \hat{\overline{K}}$ be the completion of \overline{K} , with ring of integers \mathcal{O}_C . Let $R := \varprojlim \mathcal{O}_C/p$ where the transition maps are p -th power map. R is a valuation ring with residue field \bar{k} (\bar{k} is the residue field of C). R is a perfect ring of characteristic p . Let $W(R)$ be the ring of Witt vectors. Let $\underline{\epsilon} := (\mu_{p^n})_{n=0}^\infty \in R$, $\underline{\pi} = (\pi_n)_{n=0}^\infty \in R$, and let $[\underline{\epsilon}], [\underline{\pi}]$ be their Teichmüller representatives respectively in $W(R)$.

There is a map $\theta : W(R) \rightarrow \mathcal{O}_C$ which is the unique universal lift of the map $R \rightarrow \mathcal{O}_C/p$ (projection of R onto the its first factor), and $\text{Ker } \theta$ is a principle ideal generated by $\xi = [\varpi] + p$, where $\varpi \in R$ with $\omega^{(0)} = -p$, and $[\varpi] \in W(R)$ its Teichmüller representative. Let $B_{\text{dR}}^+ := \varprojlim_n W(R)[\frac{1}{p}]/(\xi)^n$, and $B_{\text{dR}} := B_{\text{dR}}^+[\frac{1}{\xi}]$. Let $t := \log([\underline{\epsilon}])$, which is an element in B_{dR}^+ .

Let A_{cris} denote the p -adic completion of the divided power envelope of $W(R)$ with respect to $\text{Ker}(\theta)$. Let $B_{\text{cris}}^+ = A_{\text{cris}}[1/p]$ and $B_{\text{cris}} := B_{\text{cris}}^+[\frac{1}{t}]$. Let $B_{\text{st}} := B_{\text{cris}}[X]$ where X is an indeterminate. There are natural Frobenius actions, monodromy actions and filtration structures on B_{cris} and B_{st} , which we omit the definition. We have the natural embeddings $B_{\text{cris}} \subset B_{\text{st}} \subset B_{\text{dR}}$.

Let $\mathfrak{S} := W(k)[[u]]$, $E(u) \in W(k)[u]$ the minimal polynomial of π over $W(k)$, and S the p -adic completion of the PD-envelope of \mathfrak{S} with respect to the ideal $(E(u))$. We can embed the $W(k)$ -algebra $W(k)[u]$ into $W(R)$ by mapping u to $[\underline{\pi}]$. The embedding extends to the embeddings $\mathfrak{S} \hookrightarrow S \hookrightarrow A_{\text{cris}}$.

The projection from R to \bar{k} induces a projection $\nu : W(R) \rightarrow W(\bar{k})$, since $\nu(\text{Ker } \theta) = pW(\bar{k})$, the projection extends to $\nu : A_{\text{cris}} \rightarrow W(\bar{k})$, and also $\nu : B_{\text{cris}}^+ \rightarrow W(\bar{k})[\frac{1}{p}]$. Write $I_+ B_{\text{cris}}^+ := \text{Ker}(\nu : B_{\text{cris}}^+ \rightarrow W(\bar{k})[\frac{1}{p}])$, and for any subring $A \subseteq B_{\text{cris}}^+$, write $I_+ A = A \cap \text{Ker}(\nu)$.

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2. KISIN MODULES AND (φ, \hat{G}) -MODULES

In this section, we briefly review some facts in the theory of Kisin modules and (φ, \hat{G}) -modules with \mathcal{O}_E -coefficients. The materials in this section are based on works of [Kis06, Liu10, CL11, GLS14, Lev14] etc.. But here we only cite them in the form as in [Gao15, §1], where the readers can find more detailed attributions.

2.1. Kisin modules and (φ, \hat{G}) -modules with coefficients. In this subsection, all the definitions and results are valid for any finite extension K/\mathbb{Q}_p .

Recall that $\mathfrak{S} = W(k)[[u]]$ with the Frobenius endomorphism $\varphi_{\mathfrak{S}} : \mathfrak{S} \rightarrow \mathfrak{S}$ which acts on $W(k)$ via arithmetic Frobenius and sends u to u^p . Denote

$$\mathfrak{S}_{\mathcal{O}_E} := \mathfrak{S} \otimes_{\mathbb{Z}_p} \mathcal{O}_E, \quad \mathfrak{S}_{k_E} := \mathfrak{S} \otimes_{\mathbb{Z}_p} k_E = k[[u]] \otimes_{\mathbb{F}_p} k_E,$$

and extend $\varphi_{\mathfrak{S}}$ to $\mathfrak{S}_{\mathcal{O}_E}$ (resp. \mathfrak{S}_{k_E}) by acting on \mathcal{O}_E (resp. k_E) trivially. Let r be any nonnegative integer.

- Let $'\text{Mod}_{\mathfrak{S}_{\mathcal{O}_E}}^{\varphi}$ (called the category of Kisin modules of height r with \mathcal{O}_E -coefficients) be the category whose objects are $\mathfrak{S}_{\mathcal{O}_E}$ -modules \mathfrak{M} , equipped with $\varphi : \mathfrak{M} \rightarrow \mathfrak{M}$ which is a $\varphi_{\mathfrak{S}_{\mathcal{O}_E}}$ -semi-linear morphism such that the span of $\text{Im}(\varphi)$ contains $E(u)^r \mathfrak{M}$. The morphisms in the category are $\mathfrak{S}_{\mathcal{O}_E}$ -linear maps that commute with φ .
- Let $\text{Mod}_{\mathfrak{S}_{\mathcal{O}_E}}^{\varphi}$ be the full subcategory of $'\text{Mod}_{\mathfrak{S}_{\mathcal{O}_E}}^{\varphi}$ with $\mathfrak{M} \simeq \bigoplus_{i \in I} \mathfrak{S}_{\mathcal{O}_E}$ where I is a finite set.
- Let $\text{Mod}_{\mathfrak{S}_{k_E}}^{\varphi}$ be the full subcategory of $'\text{Mod}_{\mathfrak{S}_{\mathcal{O}_E}}^{\varphi}$ with $\mathfrak{M} \simeq \bigoplus_{i \in I} \mathfrak{S}_{k_E}$ where I is a finite set.

For any integer $n \geq 0$, write $n = (p-1)q(n) + r(n)$ with $q(n)$ and $r(n)$ the quotient and residue of n divided by $p-1$. Let $t^{\{n\}} = (p^{q(n)} \cdot q(n)!)^{-1} \cdot t^n$, we have $t^{\{n\}} \in A_{\text{cris}}$. We define a subring of B_{cris}^+ ,

$$\mathcal{R}_{K_0} := \left\{ \sum_{i=0}^{\infty} f_i t^{\{i\}}, f_i \in S_{K_0}, f_i \rightarrow 0 \text{ as } i \rightarrow \infty \right\}.$$

Define $\hat{\mathcal{R}} := \mathcal{R}_{K_0} \cap W(R)$. Then $\hat{\mathcal{R}}$ is a φ -stable subring of $W(R)$, which is also G_K -stable, and the G_K -action factors through \hat{G} . Denote

$$\hat{\mathcal{R}}_{\mathcal{O}_E} := \hat{\mathcal{R}} \otimes_{\mathbb{Z}_p} \mathcal{O}_E, \quad W(R)_{\mathcal{O}_E} := W(R) \otimes_{\mathbb{Z}_p} \mathcal{O}_E,$$

and extend the G_K -action and φ -action on them by acting on \mathcal{O}_E trivially. Note that $\mathfrak{S}_{\mathcal{O}_E} \subset \hat{\mathcal{R}}_{\mathcal{O}_E}$, and let $\varphi : \mathfrak{S}_{\mathcal{O}_E} \rightarrow \hat{\mathcal{R}}_{\mathcal{O}_E}$ be the composite of $\varphi_{\mathfrak{S}_{\mathcal{O}_E}} : \mathfrak{S}_{\mathcal{O}_E} \rightarrow \mathfrak{S}_{\mathcal{O}_E}$ and the embedding $\mathfrak{S}_{\mathcal{O}_E} \rightarrow \hat{\mathcal{R}}_{\mathcal{O}_E}$.

2.1.1. Definition. Let $'\text{Mod}_{\mathfrak{S}_{\mathcal{O}_E}}^{\varphi, \hat{G}}$ be the category (called the category of (φ, \hat{G}) -modules of height r with \mathcal{O}_E -coefficients) consisting of triples $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{G})$ where,

- (1) $(\mathfrak{M}, \varphi_{\mathfrak{M}}) \in '\text{Mod}_{\mathfrak{S}_{\mathcal{O}_E}}^{\varphi}$ is a Kisin module of height r ;
- (2) \hat{G} is a $\hat{\mathcal{R}}_{\mathcal{O}_E}$ -semi-linear \hat{G} -action on $\hat{\mathfrak{M}} := \hat{\mathcal{R}}_{\mathcal{O}_E} \otimes_{\varphi, \mathfrak{S}_{\mathcal{O}_E}} \mathfrak{M}$;
- (3) \hat{G} commutes with $\varphi_{\hat{\mathfrak{M}}} := \varphi_{\hat{\mathcal{R}}_{\mathcal{O}_E}} \otimes \varphi_{\mathfrak{M}}$;
- (4) Regarding \mathfrak{M} as a $\varphi(\mathfrak{S}_{\mathcal{O}_E})$ -submodule of $\hat{\mathfrak{M}}$, then $\mathfrak{M} \subseteq \hat{\mathfrak{M}}^{H_K}$;
- (5) \hat{G} acts on the $\hat{\mathfrak{M}}/(I_+ \hat{R})\hat{\mathfrak{M}}$ trivially.

A morphism between two (φ, \hat{G}) -modules is a morphism in $\text{Mod}_{\mathfrak{S}_{\mathcal{O}_E}}^{\varphi}$ which commutes with \hat{G} -actions.

We also define the following two subcategories.

- Let $\text{Mod}_{\mathfrak{S}_{\mathcal{O}_E}}^{\varphi, \hat{G}}$ be the full subcategory of $'\text{Mod}_{\mathfrak{S}_{\mathcal{O}_E}}^{\varphi, \hat{G}}$ where $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}_{\mathcal{O}_E}}^{\varphi}$.
- Let $\text{Mod}_{\mathfrak{S}_{k_E}}^{\varphi, \hat{G}}$ be the full subcategory of $'\text{Mod}_{\mathfrak{S}_{\mathcal{O}_E}}^{\varphi, \hat{G}}$ where $\mathfrak{M} \in \text{Mod}_{\mathfrak{S}_{k_E}}^{\varphi}$.

We can associate representations to (φ, \hat{G}) -modules.

2.1.2. Theorem. [Gao15, Thm. 1.2, Thm. 1.4]

- (1) Suppose $\hat{\mathfrak{M}} \in \text{Mod}_{\mathfrak{S}_{\mathcal{O}_E}}^{\varphi, \hat{G}}$ where \mathfrak{M} is of $\mathfrak{S}_{\mathcal{O}_E}$ -rank d , then

$$\hat{T}(\hat{\mathfrak{M}}) := \text{Hom}_{\hat{\mathcal{R}}, \varphi}(\hat{\mathfrak{M}}, W(R))$$

is a finite free \mathcal{O}_E -representation of G_K of rank d .

(2) Suppose $\hat{\mathfrak{M}} \in \text{Mod}_{\mathfrak{S}_{k_E}}^{\varphi, \hat{G}}$ where \mathfrak{M} is of \mathfrak{S}_{k_E} -rank d , then

$$\hat{T}(\hat{\mathfrak{M}}) := \text{Hom}_{\hat{\mathcal{R}}, \varphi}(\hat{\mathfrak{M}}, W(R) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)$$

is a finite free k_E -representation of G_K of dimension d .

(3) For $\hat{\mathfrak{M}} \in \text{Mod}_{\mathfrak{S}_{\mathcal{O}_E}}^{\varphi, \hat{G}}$, we have $\hat{T}(\hat{\mathfrak{M}}/\omega_E \hat{\mathfrak{M}}) \simeq \hat{T}(\hat{\mathfrak{M}})/\omega_E \hat{T}(\hat{\mathfrak{M}})$.

When $p > 2$, the theory of (φ, \hat{G}) -modules becomes simpler.

2.1.3. Lemma. [Gao15, Lem. 1.6] Suppose $p > 2$. Let $\hat{\mathfrak{M}} \in \text{Mod}_{\mathfrak{S}_{\mathcal{O}_E}}^{\varphi, \hat{G}}$. Then $\hat{\mathfrak{M}}$ is uniquely determined up to isomorphism by the following information:

- (1) A matrix $A_\varphi \in \text{Mat}(\mathfrak{S}_{\mathcal{O}_E})$ for the Frobenius $\varphi : \mathfrak{M} \rightarrow \mathfrak{M}$, such that there exist $B \in \text{Mat}(\mathfrak{S}_{\mathcal{O}_E})$ with $A_\varphi B = E(u)^r \text{Id}$.
- (2) A matrix $A_\tau \in \text{Mat}(\hat{\mathcal{R}}_{\mathcal{O}_E})$ (for the τ -action $\tau : \hat{\mathfrak{M}} \rightarrow \hat{\mathfrak{M}}$) such that
 - $A_\tau - \text{Id} \in \text{Mat}(I_+ \hat{\mathcal{R}}_{\mathcal{O}_E})$,
 - $A_\tau \tau(\varphi(A_\varphi)) = \varphi(A_\varphi) \varphi(A_\tau)$.
 - $g(A_\tau) = \prod_{k=0}^{\varepsilon_p(g)-1} \tau^k(A_\tau)$ for all $g \in G_\infty$ such that $\varepsilon_p(g) \in \mathbb{Z}^{\geq 0}$.

For $\overline{\mathfrak{M}} \in \text{Mod}_{\mathfrak{S}_{k_E}}^{\varphi, \hat{G}}$, it is also uniquely determined up to isomorphism by its matrix A_φ and A_τ satisfying similar conditions as above.

2.2. Rank 1 Kisin modules and (φ, \hat{G}) -modules. We only recall the following definitions and results in the \mathbb{Q}_p case.

2.2.1. Definition.

- (1) Suppose t is a non-negative integer, $a \in k_E^\times$. Let $\overline{\mathfrak{M}}(t; a)$ be the rank-1 module in $\text{Mod}_{\mathfrak{S}_{k_E}}^\varphi$ such that $\overline{\mathfrak{M}}(t; a)$ is generated by some basis e , and $\varphi(e) = au^t e$.
- (2) Suppose t is a non-negative integer, $\hat{a} \in \mathcal{O}_E^\times$. Let $\mathfrak{M}(t; \hat{a})$ be the rank-1 module in $\text{Mod}_{\mathcal{O}_E}^\varphi$ such that $\mathfrak{M}(t; \hat{a})$ is generated by some basis \tilde{e} , and $\varphi(\tilde{e}) = \hat{a}(u - p)^t \tilde{e}$.

2.2.2. Lemma. [Gao15, Lem. 1.11]

- (1) Any rank 1 module in $\text{Mod}_{\mathfrak{S}_{k_E}}^\varphi$ is of the form $\overline{\mathfrak{M}}(t; a)$ for some t and a .
- (2) When \hat{a} is a lift of a , $\mathfrak{M}(t; \hat{a})/\omega_E \mathfrak{M}(t; \hat{a}) \simeq \overline{\mathfrak{M}}(t; a)$.
- (3) There is a unique $\hat{\mathfrak{M}}(t; \hat{a}) \in \text{Mod}_{\mathfrak{S}_{\mathcal{O}_E}}^{\varphi, \hat{G}}$ such that the ambient Kisin module of $\hat{\mathfrak{M}}(t; \hat{a})$ is $\mathfrak{M}(t; \hat{a})$, and $\hat{T}(\hat{\mathfrak{M}}(t; \hat{a}))$ is a crystalline character. In fact, $\hat{T}(\hat{\mathfrak{M}}(t; \hat{a})) = \lambda_{\hat{a}} \psi^t$, where ψ is a certain crystalline character such that $\text{HT}(\psi) = 1$, and $\lambda_{\hat{a}}$ is the unramified character of $G_{\mathbb{Q}_p}$ which sends the arithmetic Frobenius to \hat{a} .
- (4) There is a unique $\overline{\mathfrak{M}}(t; a) \in \text{Mod}_{\mathfrak{S}_{k_E}}^{\varphi, \hat{G}}$ such that the ambient Kisin module is $\overline{\mathfrak{M}}(t; a)$. Furthermore, $\hat{T}(\overline{\mathfrak{M}}(t; a))$ is the reduction of $\hat{T}(\hat{\mathfrak{M}}(t; \hat{a}))$ for any lift $\hat{a} \in \mathcal{O}_E$ of a .

3. SOME GROUP THEORY

We recall some group theory, which will be useful for our work. All the materials in this section are developed in [BH15, §2.3], for general split connected reductive groups. But we will only need it for GL_d , which we recall.

Let H be the algebraic group GL_d , T the torus consisting of diagonal matrices, B the Borel consisting of upper triangular matrices, and U the unipotent radical consisting of unipotent matrices.

We have $X(T) := \mathrm{Hom}_{\mathrm{alg}}(T, \mathbb{G}_m) = \mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_d$, where ϵ_i is the character sending the diagonal matrix $[x_1, \dots, x_d]$ to x_i . Let $S = \{\epsilon_i - \epsilon_{i+1} : 1 \leq i \leq d-1\}$ be the simple roots, and let $R^+ = \{\epsilon_i - \epsilon_j : 1 \leq i < j \leq d\}$ be the positive roots. Denote W the Weyl group of H , which is isomorphic to the permutation group S_d . If $\alpha = \epsilon_i - \epsilon_j \in R^+$, let $U_\alpha \subset H$ be the root subgroup, which corresponds to the unipotent upper triangular matrices where the only nonzero element above the diagonal is at the (i, j) -position.

3.0.3. Definition. A subset $C \subseteq R^+$ is called *closed* if the following condition is satisfied: if $\alpha \in C, \beta \in C$ and $\alpha + \beta \in R^+$, then $\alpha + \beta \in C$.

For a closed subset $C \subseteq R^+$, let $U_C \subseteq U$ be the Zariski closed subgroup of B generated by the subgroups U_α for all $\alpha \in C$. Let $B_C = TU_C \subseteq B$. If $C = \{\epsilon_{i_1} - \epsilon_{j_1}, \dots, \epsilon_{i_m} - \epsilon_{j_m}\}$ is a closed subset of R^+ , then it is easy to see that B_C corresponds to the matrices where the only nonzero elements above the diagonal are at the positions (i_ℓ, j_ℓ) for all $1 \leq \ell \leq m$.

Recall that if we let $N_H(T)$ be the normalizer of T in H , then $N_H(T)/T$ is isomorphic to W . For each $\sigma \in W$ which is a permutation sending $(1, \dots, d)$ to $(\sigma(1), \dots, \sigma(d))$, we fix a representative of σ in H to be the $d \times d$ matrix $w_\sigma := (\delta_{i, \sigma(j)})_{1 \leq i, j \leq d} = (\delta_{\sigma^{-1}(i), j})$ where the notation $\delta_{x, y} = 0$ if $x \neq y$, and $\delta_{x, y} = 1$ if $x = y$. Note that if we have another $d \times d$ matrix $A = (a_{k, l})$, we have the matrix multiplication:

$$(\delta_{i, \sigma(j)})(a_{k, l}) = (a_{\sigma^{-1}(k), l}), \quad (a_{k, l})(\delta_{i, \sigma(j)}) = (a_{k, \sigma(l)}),$$

and so in particular $\sigma^{-1}(a_{i, j})\sigma = (a_{\sigma(i), \sigma(j)})$.

Let $C \subseteq R^+$ closed, we define the following subset of W :

$$W_C := \{\sigma \in W : \sigma^{-1}(C) \subseteq R^+\}.$$

3.0.4. Lemma. [BH15, Lem. 2.3.6] *With notations above, we have*

$$W_C = \{\sigma \in W : w_\sigma^{-1}B_Cw_\sigma \subseteq B\}.$$

The above lemma says that conjugations of a matrix in B_C by permutations in W_C will stay upper triangular. In the following, we will sometimes simply use σ to mean the matrix w_σ . If \cdot is a ring, we will use $B_C(\cdot)$ to mean the subgroup of $\mathrm{Mat}_d(\cdot)$ corresponding to the algebraic group B_C .

4. SHAPE OF UPPER TRIANGULAR TORSION (φ, \hat{G}) -MODULES

In this section, we study the shape of upper triangular torsion (φ, \hat{G}) -modules, using results in [Gao15], as well as ideas in Section 3.

4.1. Shape of φ .

4.1.1. Proposition. *With notations from (CRYS). Suppose that $\bar{\rho}$ is upper triangular. Then $\overline{\mathfrak{M}} \in \mathcal{E}(\overline{\mathfrak{N}}_d, \dots, \overline{\mathfrak{N}}_1)$, where $\overline{\mathfrak{N}}_i = \overline{\mathfrak{M}}(t_i; a_i)$ for some $a_i \in k_E^\times$, and $\{t_1, \dots, t_d\} = \{r_1, \dots, r_d\}$ as sets.*

Furthermore, there exists a basis \mathbf{e} of $\overline{\mathfrak{M}}$, such that the matrix A_φ of φ with respect to this basis can be decomposed as $A_\varphi = \widetilde{A}_\varphi + u^p N$ where

- (1) \widetilde{A}_φ is upper triangular, with diagonal equal to $[a_1 u^{t_1}, \dots, a_d u^{t_d}]$, and $(\widetilde{A}_\varphi)_{i,j} = u^{t_i} y_{i,j}$ for $i < j$ (here $(\widetilde{A}_\varphi)_{i,j}$ is the element of \widetilde{A}_φ in the (i,j) -position), where
- $y_{i,j} = 0$ if $t_j < t_i$.
 - $y_{i,j} \in k_E$ if $t_j > t_i$.
- (2) $N \in \text{Mat}(k_E[u])$ is strictly upper triangular (i.e., the diagonal is 0).

Proof. This is a slight generalization of [Gao15, Prop. 4.1]. That is, we can allow the existence of nonzero morphisms $\overline{\mathfrak{M}}_j \rightarrow \overline{\mathfrak{M}}_i$ for some $j > i$, i.e., the situation in Statement (3) of [Gao15, Prop. 2.2] is allowed.

The existence and shape of $\overline{\mathfrak{M}}_i$ is proved in [Gao15, Prop. 2.3]. The existence of the basis \mathbf{e} and the upper triangular matrix A_φ follows from [Gao15, Prop. 2.2]. Recall that in *loc. cit.*, when there exists nonzero morphisms $\overline{\mathfrak{M}}_j \rightarrow \overline{\mathfrak{M}}_i$ for some $j > i$, then A_φ can have *extra* terms as described in Statement (3) of *loc. cit.*, and this extra term has degree $t_j + \frac{t_j - t_i}{p-1}$. Note that in order to have $\overline{\mathfrak{M}}_j \rightarrow \overline{\mathfrak{M}}_i$ for $j > i$, the only possibility is to have $t_j - t_i = p - 1$ and $a_i = a_j$ (easy by [Gao15, Lem. 1.13] since we are in the \mathbb{Q}_p situation). So the extra terms are always of degree p or $p + 1$, i.e., the extra terms are always divisible by u^p . (In fact, clearly we can only have at most two extra terms.) Decompose A_φ as $\widetilde{A}_\varphi + u^p N$ where $u^p N$ are the extra terms.

Now argue similarly as in [Gao15, Prop. 2.3], let \mathbf{e}' be another basis of $\overline{\mathfrak{M}}$ such that $\varphi(\mathbf{e}') = \mathbf{e}' X[u^{r_1}, \dots, u^{r_d}]$ where $X \in \text{Mat}_d(k_E[[u]])$ by [Gao15, Thm. 2.1]. Let $\mathbf{e}' = \mathbf{e}T$ for some matrix $T \in \text{Mat}_d(k_E[[u]])$, then $A_\varphi = TX[u^{r_1}, \dots, u^{r_d}]\varphi(T^{-1})$. Similarly as in [Gao15, Prop. 4.1], let $\varphi(T) = P + u^p Q$ for some $P \in \text{Mat}_d(k_E)$, $Q \in \text{Mat}_d(k_E[[u]])$, and let $R \in \text{GL}_d(k_E)$ such that $R^{-1}[u^{r_1}, \dots, u^{r_d}]R = [u^{t_1}, \dots, u^{t_d}]$, then we have

$$(\widetilde{A}_\varphi + u^p N)(P + u^p Q)R = TXR[u^{t_1}, \dots, u^{t_d}].$$

So we have $u^{t_i} \mid \text{col}_i(\widetilde{A}_\varphi PR)$. Now we can again apply [Gao15, Lem. 4.3] to conclude (note that \widetilde{A}_φ satisfies property (DEG) of *loc. cit.*, since we removed the extra terms $u^p N$ from A_φ).

□

With notations in Proposition 4.1.1, we can define the following subset C of R^+ :

$$(4.1.2) \quad C := \{\epsilon_i - \epsilon_j : i < j, t_i < t_j\}.$$

It is easy to see that C is closed in R^+ , and A_φ is a matrix in the subgroup $B_C(k_E[u])$.

4.1.3. Proposition. *There exists a unique $\sigma \in W_C$ such that $\sigma^{-1}A_\varphi\sigma$ is still upper triangular, and $\text{diag}(\sigma^{-1}A_\varphi\sigma) = [a_{\sigma(1)}u^{r_1}, \dots, a_{\sigma(d)}u^{r_d}]$.*

Proof. The uniqueness of σ is determined since we have $t_{\sigma(i)} = r_i, \forall i$, that is,

$$(4.1.4) \quad t_{\sigma(1)} < \dots < t_{\sigma(d)}$$

It suffices to show that $\sigma \in W_C$ ($\Leftrightarrow \sigma^{-1}(W_C) \subseteq R^+$), i.e., if $\epsilon_i - \epsilon_j \in C$, then $\sigma^{-1}(i) < \sigma^{-1}(j)$. Let $x = \sigma^{-1}(i)$ and $\sigma^{-1}(j) = y$. Then $t_i = t_{\sigma(x)} < t_{\sigma(y)} = t_j$. So by (4.1.4), we must have $x < y$. □

4.2. Shape of τ . Our following lemma (Lemma 4.2.1) is valid for any K/\mathbb{Q}_p . So we use notations introduced in Section 1. Recall that $u = (\pi_n)_{n=0}^\infty \in R$, and we normalize the valuation on R so that $v_R(u) = \frac{1}{e}$ where e is the ramification degree of K/\mathbb{Q}_p . For $\zeta \in R \otimes_{\mathbb{F}_p} k_E$, write it as $\zeta = \sum_{i=1}^m y_i \otimes a_i$ where $y_i \in R$, and $a_i \in k_E$ are independent over \mathbb{F}_p . Let

$$v_R(\zeta) := \min\{v_R(y_i)\}.$$

Then by [Gao15, Lem. 5.6], v_R is a well-defined valuation on $R \otimes_{\mathbb{F}_p} k_E$ (so in particular, it does not depend on the sum representing ζ). In particular, $v_R(\varphi(\zeta)) = pv_R(\zeta)$. We also use the convention that $v_R(0) = +\infty$.

4.2.1. Lemma. *Let $\zeta \in R \otimes_{\mathbb{F}_p} k_E$ with $v_R(\zeta) > 0$, such that*

$$\zeta \tau(\varphi(u^b)) = \varphi(u^a) \varphi(\zeta)$$

for some $a > b \geq 0$, then $\zeta = 0$.

Proof. Note that $\tau(u) = u\epsilon$, where $\epsilon = (\mu_{p^n})_{n=0}^\infty \in R$. Consider the valuation on both side of the equation, then $v_R(\zeta) + \frac{pb}{e} = \frac{pa}{e} + pv_R(\zeta)$. The only possibility is when $v_R(\zeta) = +\infty$. \square

Now we return to the \mathbb{Q}_p case.

4.2.2. Proposition. *With notations in Proposition 4.1.1, suppose $A_\tau \in \text{Mat}(R \otimes_{\mathbb{F}_p} k_E)$ is the matrix of τ with respect to the basis $1 \otimes_\varphi e$, then A_τ is in the subgroup $B_C(R \otimes_{\mathbb{F}_p} k_E)$ defined by (4.1.2), i.e., if $i < j$ and $t_i > t_j$, then $(A_\tau)_{i,j} = 0$.*

Proof. This is easy consequence of the following lemma. Note that for any $i < j$, $v_R((A_\tau)_{i,j}) > 0$ by [Gao15, Lem. 5.7]. \square

4.2.3. Lemma. *Let $F = (f_{i,j}) \in \text{Mat}_d(k_E[[u]])$, $M = (m_{i,j}) \in \text{Mat}_d(R \otimes_{\mathbb{F}_p} k_E)$ two upper triangular matrices. Suppose $\text{diag}(F) = [a_1 u^{t_1}, \dots, a_d u^{t_d}]$ where $a_i \in k_E^\times$ and t_i are distinct non-negative integers. Suppose that*

- If $i < j$ and $t_i > t_j$, then $f_{i,j} = 0$,
- $v_R(m_{i,j}) > 0, \forall i < j$, and
- $M\tau(\varphi(F)) = \varphi(F)\varphi(M)$.

Then $M \in B_C(R \otimes_{\mathbb{F}_p} k_E)$, where $C := \{\epsilon_i - \epsilon_j : i < j, t_i < t_j\}$ is a closed subset of R^+ .

Proof. We prove by induction on the dimension d . When $d = 1$, there is nothing to prove. Suppose the lemma is true for dimension less than d , and consider it for d .

We can apply the induction hypothesis to $F_{1,1}$ and $M_{1,1}$ (resp. $F_{d,d}$ and $M_{d,d}$), where $F_{1,1}$ is the co-matrix of F by deleting the 1st row and 1st column (and similarly for $M_{1,1}, F_{d,d}$ and $M_{d,d}$). So we only need to deal with the element on the most upper right corner. That is, we only need to prove that if $t_1 > t_d$, then $m_{1,d} = 0$.

For any $2 \leq i \leq d$, we have

- (Case 1) If $t_i > t_1 > t_d$, then $f_{i,d} = 0$ (property of F), and $m_{i,d} = 0$ (induction hypothesis).
- (Case 2) If $t_1 > t_i$, then $f_{1,i} = 0$ (property of F), and $m_{1,i} = 0$ (induction hypothesis).

By the condition that $M\tau(\varphi(F)) = \varphi(F)\varphi(M)$, we must have

$$\sum_{i=1}^d m_{1,i}\tau(\varphi(f_{i,d})) = \sum_{i=1}^d \varphi(f_{1,i})\varphi(m_{i,d}).$$

So we will always have $m_{1,d}\tau(\varphi(u^{t_d})) = \varphi(u^{t_1})\varphi(m_{1,d})$, because all the other terms vanish. Now we can conclude $m_{1,d} = 0$ by Lemma 4.2.1. \square

5. CRYSTALLINE LIFTING THEOREM

5.0.4. Theorem. *With notations in (CRYS), and suppose that $\bar{\rho}$ is upper triangular. Suppose $\bar{\rho} \in \mathcal{E}(\bar{\chi}_1, \dots, \bar{\chi}_d)$ such that $\bar{\chi}_i \bar{\chi}_j^{-1} \neq \bar{\varepsilon}_p, \forall i \neq j$. Then there exists an upper triangular crystalline representation ρ' such that $\bar{\rho}' \cong \bar{\rho}$, and $\text{HT}(\rho') = \text{HT}(\rho)$ as sets.*

Proof. Recall that $e = (e_1, \dots, e_d)$ is the basis of $\bar{\mathfrak{M}}$ in Proposition 4.1.1. Let $\sigma \in W_C$ be the unique element as in Proposition 4.1.3, and denote $e^\sigma := (e_{\sigma(1)}, \dots, e_{\sigma(d)})$. By *loc. cit.*, the matrix of φ for $\bar{\mathfrak{M}}$ with respect to the basis e^σ (which is $\sigma^{-1}A_\varphi\sigma$) is still upper triangular. By Proposition 4.2.2 and Lemma 3.0.4, the matrix of τ for $\bar{\mathfrak{M}}$ with respect to the basis $1 \otimes_\varphi e^\sigma$ (which is $\sigma^{-1}A_\tau\sigma$) is also upper triangular. That is to say (by Lemma 2.1.3), $\bar{\mathfrak{M}} \in \mathcal{E}(\bar{\mathfrak{M}}_{\sigma(d)}, \dots, \bar{\mathfrak{M}}_{\sigma(1)})$, where $\bar{\mathfrak{M}}_{\sigma(i)} := \bar{\mathfrak{M}}(r_i; a_{\sigma(i)})$. And so $\bar{\rho} = \hat{T}(\bar{\mathfrak{M}}) \in \mathcal{E}(\bar{\chi}_{\sigma(1)}, \dots, \bar{\chi}_{\sigma(d)})$.

By [Gao15, Lem. 1.11(3)], each $\bar{\chi}_{\sigma(i)}$ has a crystalline lift $\chi_{\sigma(i)} =: \hat{T}(\hat{\mathfrak{M}}(r_i; \hat{a}_{\sigma(i)}))$ where $\hat{a}_{\sigma(i)} \in \mathcal{O}_E^\times$ is any lift of $a_{\sigma(i)}$. Since $r_1 < \dots < r_d$, by [GG12, Lem. 3.1.5] (note that our convention of Hodge-Tate weights is the opposite of *loc. cit.*), $\bar{\rho}$ has an upper triangular crystalline lift ρ' such that $\rho' \in \mathcal{E}(\chi_{\sigma(1)}, \dots, \chi_{\sigma(d)})$. Remark that ρ' is in fact *ordinary* in the sense of [GG12, Def. 3.1.3]. \square

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